ESTIMATION OF THE COHERENCE FUNCTION WITH THE MVDR APPROACH

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ABSTRACT

The minimum variance distortionless response (MVDR), originally developed by Capon for frequency-wavenumber analysis, is a very well established method in array processing. It is also used in spectral estimation. The aim of this paper is to show how the MVDR method can be used to estimate the magnitude squared coherence (MSC) function, which is very useful in so many applications but so few methods exist to estimate it. Simulations show that our algorithm gives much more reliable results than the one based on the popular Welch's method.

1. INTRODUCTION

Spectral estimation plays a major role in signal processing. It has numerous applications in diversified fields such as radar, sonar, speech, communications, biomedical, etc [1], [2], [3]. There are basically two broad categories of techniques for spectral estimation. One is the non-parametric approach, which is based on the concept of bandpass filtering. The other is the parametric method, which assumes a model for the data, and the spectral estimation then becomes a problem of estimating the parameters in the assumed model. If the model fits the data well, the latter may yield more accurate spectral estimate than the former. However, in the case that the model does not satisfy the data, the parametric model will suffer significant performance degradation and lead to a biased estimate. Therefore, a great deal of research efforts are still devoted to the nonparametric approaches.

One of the most well-known non-parametric spectral estimation algorithms is the Capon's approach, which is also known as minimum variance distortionless response (MVDR) [4], [5]. This technique was extensively studied in the literature and is considered as a high-resolution method. The MVDR spectrum can be viewed as the output of a bank of filters, with each filter centered at one of the analysis frequencies. Its bandpass filters are both data and frequency dependent which is the main difference with a periodogram-based approach where its bandpass filters are a discrete Fourier matrix, which is both data and frequency independent [3], [6].

The objective of this paper is to generalize the concept of the MVDR spectrum to cross-spectrum estimation and most importantly, to show how to use this approach to estimate the magnitude squared coherence (MSC) function as an alternative to the popular Welch's method [7], [8].

2. THE MVDR SPECTRUM

Capon's method for spectral estimation is based on a filter-bank decomposition: the spectrum of a signal is estimated in each band by a simple filter design subject to some constraints [4], [5].

Let $x(n)$ be a zero-mean stationary random process which is the input of K filters of length L,

$$\begin{aligned}\mathbf{g}_k &= \left[\begin{array}{cccc} g_{k,0} & g_{k,1} & \cdots & g_{k,L-1}\end{array}\right]^T, \\ k &= 0, 1, \cdots, K-1,\end{aligned}$$

where superscript ^{T} denotes transposition.

If we denote by $y_k(n)$ the output signal of the filter $\mathbf{g}_k$, its power is:

$$\begin{aligned}E\left\{|y_k(n)|^2\right\} &= E\left\{\left|\mathbf{g}_k^H \mathbf{x}(n)\right|^2\right\} \\ &= \mathbf{g}_k^H \mathbf{R}_{xx} \mathbf{g}_k, \end{aligned} \tag{1}$$

where $E\{\cdot\}$ is the mathematical expectation, superscript ^{H} denotes transpose conjugate of a vector or a matrix,

$$\mathbf{R}_{xx} = E\left\{\mathbf{x}(n)\mathbf{x}^H(n)\right\} \tag{2}$$

is the covariance matrix of the input signal $x(n)$, and

$$\mathbf{x}(n) \quad = \quad \left[\begin{array}{cccc} x(n) & x(n-1) & \cdots & x(n-L+1) \end{array}\right]^T .$$

In the rest of this paper, we always assume that $\mathbf{R}_{xx}$ is positive definite.

Consider the $(L \times K)$ matrix,

$$\mathbf{F} = \left[\begin{array}{cccc} \mathbf{f}_0 & \mathbf{f}_1 & \cdots & \mathbf{f}_{K-1} \end{array}\right],$$

where

$$\mathbf{f}_k = \frac{1}{\sqrt{L}} \left[\begin{array}{cccc} 1 & \exp(j\omega_k) & \cdots & \exp(j\omega_k(L-1)) \end{array}\right]^T$$

and $\omega_k = 2\pi k/K,\ k = 0, 1, \cdots, K-1$. For $K = L$, $\mathbf{F}$ is called the Fourier matrix and is unitary, i.e. $\mathbf{F}^H\mathbf{F} = \mathbf{F}\mathbf{F}^H = \mathbf{I}$. In the MVDR spectrum, the filter coefficients are chosen so as to minimize the variance of the filter output, subject to the constraint:

$$\mathbf{g}_k^H \mathbf{f}_k = \mathbf{f}_k^H \mathbf{g}_k = 1. \tag{3}$$

Under this constraint, the process $x(n)$ is passed through the filter $\mathbf{g}_k$ with no distortion at frequency ω_k and signals at other frequencies than ω_k tend to be attenuated. Mathematically, this is equivalent to minimizing the following cost function:

$$J_k = \mathbf{g}_k^H \mathbf{R}_{xx} \mathbf{g}_k + \mu \left[1 - \mathbf{g}_k^H \mathbf{f}_k\right], \tag{4}$$

where μ is a Lagrange multiplier. The minimization of (4) leads to the following solution:

$$\mathbf{g}_k = \frac{\mathbf{R}_{xx}^{-1} \mathbf{f}_k}{\mathbf{f}_k^H \mathbf{R}_{xx}^{-1} \mathbf{f}_k}. \tag{5}$$

We define the spectrum of $x(n)$ at ω_k as,

$$S_{xx}(\omega_k) = E\left\{|y_k(n)|^2\right\} = \mathbf{g}_k^H \mathbf{R}_{xx} \mathbf{g}_k. \tag{6}$$

Therefore, plugging (5) into (6), we find that:

$$S_{xx}(\omega_k) = \frac{1}{\mathbf{f}_k^H \mathbf{R}_{xx}^{-1} \mathbf{f}_k}. \tag{7}$$

Replacing the previous equation in (5), we get:

$$\mathbf{R}_{xx} \mathbf{g}_k = S_{xx}(\omega_k) \mathbf{f}_k. \tag{8}$$

Taking into account all vectors $\mathbf{f}_k$, $k = 0, 1, \cdots, K-1$, (8) has the general form:

$$\mathbf{R}_{xx} \mathbf{G} = \mathbf{F} \mathbf{S}_{xx}(\omega), \tag{9}$$

where

$$\mathbf{G} = \left[\begin{array}{cccc} \mathbf{g}_0 & \mathbf{g}_1 & \cdots & \mathbf{g}_{K-1} \end{array}\right]$$

and

$$\mathbf{S}_{xx}(\omega) = \text{diag}\left\{S_{xx}(\omega_0), S_{xx}(\omega_1), \cdots, S_{xx}(\omega_{K-1})\right\}$$

is a diagonal matrix.

3. APPLICATION TO THE CROSS-SPECTRUM AND MAGNITUDE SQUARED COHERENCE FUNCTION

In this section, we show how to use the MVDR approach for the estimation of the cross-spectrum and the magnitude squared coherence function.

3.1. An MVDR Cross-Spectrum

We assume here that we have two zero-mean stationary random signals $x_1(n)$ and $x_2(n)$ with respective spectra $S_{x_1x_1}(\omega_k)$ and $S_{x_2x_2}(\omega_k)$. As explained in Section 2, we can design two filters,

$$\mathbf{g}_{p,k} = \frac{\mathbf{R}_{x_px_p}^{-1} \mathbf{f}_k}{\mathbf{f}_k^H \mathbf{R}_{x_px_p}^{-1} \mathbf{f}_k},\ p = 1, 2, \tag{10}$$

to find the spectra of $x_1(n)$ and $x_2(n)$ at frequency ω_k:

$$S_{x_px_p}(\omega_k) = \frac{1}{\mathbf{f}_k^H \mathbf{R}_{x_px_p}^{-1} \mathbf{f}_k},\ p = 1, 2, \tag{11}$$

where

$$\mathbf{R}_{x_px_p} = E\left\{\mathbf{x}_p(n)\mathbf{x}_p^H(n)\right\} \tag{12}$$

is the covariance matrix of the signal $x_p(n)$ and

$$\mathbf{x}_p(n) = \left[\begin{array}{cccc} x_p(n) & x_p(n-1) & \cdots & x_p(n-L+1) \end{array}\right]^T.$$

Let $y_{1,k}(n)$ and $y_{2,k}(n)$ be the respective outputs of the filters $\mathbf{g}_{1,k}$ and $\mathbf{g}_{2,k}$. We define the cross-spectrum between $x_1(n)$ and $x_2(n)$ at frequency ω_k as,

$$S_{x_1x_2}(\omega_k) = E\left\{y_{1,k}(n) y_{2,k}^*(n)\right\}, \tag{13}$$

where the superscript * is the complex conjugate operator. Similarly,

$$\begin{aligned} S_{x_2x_1}(\omega_k) &= E\left\{y_{2,k}(n) y_{1,k}^*(n)\right\} \\ &= S_{x_1x_2}^*(\omega_k). \end{aligned} \tag{14}$$

Now if we develop (13), we get:

$$S_{x_1x_2}(\omega_k) = \mathbf{g}_{1,k}^H \mathbf{R}_{x_1x_2} \mathbf{g}_{2,k}, \tag{15}$$

where

$$\mathbf{R}_{x_1x_2} = E\left\{\mathbf{x}_1(n)\mathbf{x}_2^H(n)\right\} \tag{16}$$

is the cross-correlation matrix between $x_1(n)$ and $x_2(n)$. Replacing (10) in (15), we obtain the cross-spectrum:

$$S_{x_1x_2}(\omega_k) = \frac{\mathbf{f}_k^H \mathbf{R}_{x_1x_1}^{-1} \mathbf{R}_{x_1x_2} \mathbf{R}_{x_2x_2}^{-1} \mathbf{f}_k}{\left[\mathbf{f}_k^H \mathbf{R}_{x_1x_1}^{-1} \mathbf{f}_k\right] \left[\mathbf{f}_k^H \mathbf{R}_{x_2x_2}^{-1} \mathbf{f}_k\right]}. \tag{17}$$

3.2. An MVDR Magnitude Squared Coherence Function

We define the magnitude squared coherence (MSC) function between two signals $x_1(n)$ and $x_2(n)$ as,

$$\gamma^2_{x_1x_2}(\omega_k) = \frac{|S_{x_1x_2}(\omega_k)|^2}{S_{x_1x_1}(\omega_k)S_{x_2x_2}(\omega_k)}. \qquad (18)$$

From (17), we deduce the magnitude squared cross-spectrum:

$$|S_{x_1x_2}(\omega_k)|^2 = \frac{\left|\mathbf{f}_k^H\mathbf{R}_{x_1x_1}^{-1}\mathbf{R}_{x_1x_2}\mathbf{R}_{x_2x_2}^{-1}\mathbf{f}_k\right|^2}{\left[\mathbf{f}_k^H\mathbf{R}_{x_1x_1}^{-1}\mathbf{f}_k\right]^2\left[\mathbf{f}_k^H\mathbf{R}_{x_2x_2}^{-1}\mathbf{f}_k\right]^2}. \qquad (19)$$

Using expressions (11) and (19) in (18), the MSC becomes:

$$\gamma^2_{x_1x_2}(\omega_k) = \frac{\left|\mathbf{f}_k^H\mathbf{R}_{x_1x_1}^{-1}\mathbf{R}_{x_1x_2}\mathbf{R}_{x_2x_2}^{-1}\mathbf{f}_k\right|^2}{\left[\mathbf{f}_k^H\mathbf{R}_{x_1x_1}^{-1}\mathbf{f}_k\right]\left[\mathbf{f}_k^H\mathbf{R}_{x_2x_2}^{-1}\mathbf{f}_k\right]}. \qquad (20)$$

Property: We have,

$$0 \le \gamma^2_{x_1x_2}(\omega_k) \le 1, \ \forall \mathbf{f}_k. \qquad (21)$$

Proof: Since matrices $\mathbf{R}_{x_1x_1}$ and $\mathbf{R}_{x_2x_2}$ are assumed to be positive definite, it is clear that $\gamma^2_{x_1x_2}(\omega_k) \ge 0$. To prove that $\gamma^2_{x_1x_2}(\omega_k) \le 1$, we need to rewrite the MSC function. Define the vectors,

$$\mathbf{f}_{p,k} = \mathbf{R}_{x_px_p}^{-1/2}\mathbf{f}_k, \ p = 1, 2, \qquad (22)$$

and the normalized cross-correlation matrix,

$$\mathbf{R}_{\mathrm{n},x_1x_2} = \mathbf{R}_{x_1x_1}^{-1/2}\mathbf{R}_{x_1x_2}\mathbf{R}_{x_2x_2}^{-1/2}. \qquad (23)$$

Using the previous definitions in (20), the MSC is now:

$$\gamma^2_{x_1x_2}(\omega_k) = \frac{\left|\mathbf{f}_{1,k}^H\mathbf{R}_{\mathrm{n},x_1x_2}\mathbf{f}_{2,k}\right|^2}{\left[\mathbf{f}_{1,k}^H\mathbf{f}_{1,k}\right]\left[\mathbf{f}_{2,k}^H\mathbf{f}_{2,k}\right]}. \qquad (24)$$

Consider the Hermitian positive semi-definite matrix,

$$\mathbf{M} = \begin{bmatrix} \mathbf{I} & \mathbf{R}_{\mathrm{n},x_1x_2} \\ \mathbf{R}_{\mathrm{n},x_1x_2}^H & \mathbf{I} \end{bmatrix}, \qquad (25)$$

and the vectors

$$\mathbf{f}'_{1,k} = \begin{bmatrix} \mathbf{f}_{1,k} \\ \mathbf{0} \end{bmatrix}, \qquad (26)$$

$$\mathbf{f}'_{2,k} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_{2,k} \end{bmatrix}. \qquad (27)$$

We can easily check that,

$$\left|\mathbf{f}'^H_{1,k}\mathbf{M}\mathbf{f}'_{2,k}\right|^2 = \left|\mathbf{f}_{1,k}^H\mathbf{R}_{\mathrm{n},x_1x_2}\mathbf{f}_{2,k}\right|^2, \qquad (28)$$

$$\mathbf{f}'^H_{p,k}\mathbf{M}\mathbf{f}'_{p,k} = \mathbf{f}_{p,k}^H\mathbf{f}_{p,k}, \ p = 1, 2. \qquad (29)$$

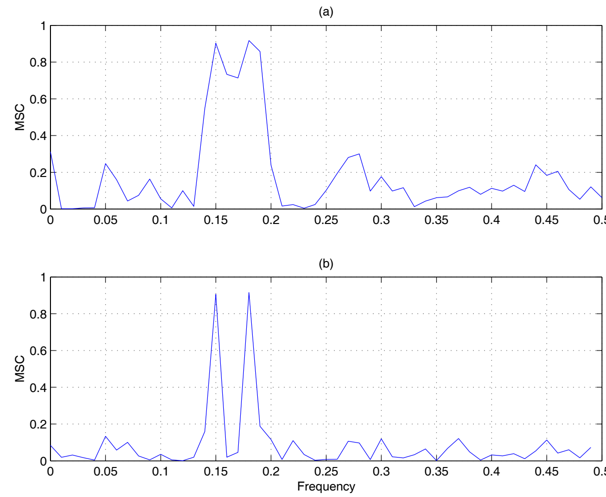

Fig. 1. Estimation of the magnitude squared coherence function. (a) MATLAB function 'cohere' with a window length of 100. (b) Proposed algorithm with $K = L = 100$. Conditions of simulations: 2 coherent frequencies at $\nu_0 = 0.15$ and $\nu_1 = 0.18$.

Inserting these expressions in the Cauchy-Schwartz inequality,

$$\left|\mathbf{f}'^H_{1,k}\mathbf{M}\mathbf{f}'_{2,k}\right|^2 \le \left[\mathbf{f}'^H_{1,k}\mathbf{M}\mathbf{f}'_{1,k}\right]\left[\mathbf{f}'^H_{2,k}\mathbf{M}\mathbf{f}'_{2,k}\right], \qquad (30)$$

we see that $\gamma^2_{x_1x_2}(\omega_k) \le 1, \ \forall \mathbf{f}_k$.

This property was, of course, expected in order that the definition (20) of the MSC could have a sense.

4. SIMULATIONS

In this section, we compare by way of simulations, the performance of the MSC function estimated with our approach and with the MATLAB function 'cohere' which uses the Welch's averaged periodogram method [7], [8]. We consider the illustrative example of two signals $x_1(n)$ and $x_2(n)$ which don't have that much in common except for N_f sinusoids at frequencies $\nu_0, \nu_1, \cdots, \nu_{N_\mathrm{f}-1}$,

$$x_1(n) = w_1(n) + \sum_{i=0}^{N_\mathrm{f}-1} \cos(2\pi\nu_i n), \qquad (31)$$

$$x_2(n) = w_2(n) + \sum_{i=0}^{N_\mathrm{f}-1} \cos[2\pi(\nu_i n + \phi_i)], \qquad (32)$$

where $w_1(n)$ and $w_2(n)$ are two independent zero-mean (real) white Gaussian random processes with unit variance. The phases $\phi_0, \phi_1, \cdots, \phi_{N_\mathrm{f}-1}$, in the signal $x_2(n)$ are random. In this example, the theoretical coherence should be equal to 1 at frequencies $\nu_0, \nu_1, \cdots, \nu_{N_\mathrm{f}-1}$, and 0 at the others. For both algorithms, we worked on 1024 time samples.

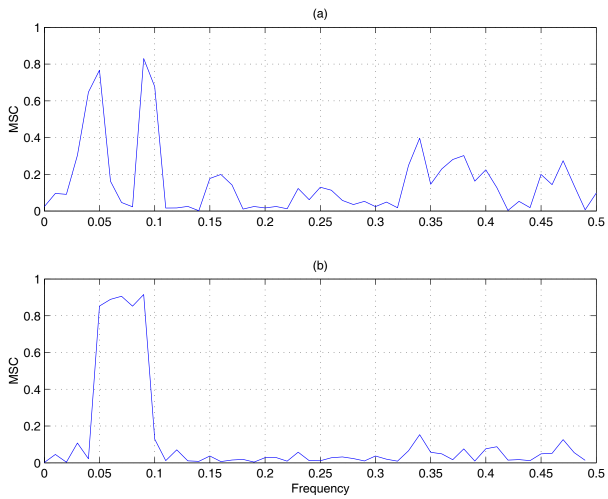

Fig. 2. Estimation of the magnitude squared coherence function. (a) MATLAB function 'cohere' with a window length of 100. (b) Proposed algorithm with $K = L = 100$. Conditions of simulations: 5 coherent frequencies at $\nu_0 = 0.05$, $\nu_1 = 0.06$, $\nu_2 = 0.07$, $\nu_3 = 0.08$, and $\nu_4 = 0.09$.

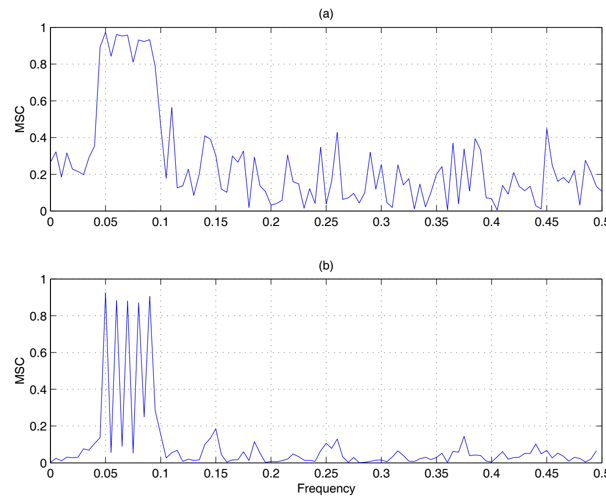

Fig. 3. Estimation of the magnitude squared coherence function. (a) MATLAB function 'cohere' with a window length of 200. (b) Proposed algorithm with $K = 200$ and $L = 100$. Conditions of simulations: 5 coherent frequencies at $\nu_0 = 0.05$, $\nu_1 = 0.06$, $\nu_2 = 0.07$, $\nu_3 = 0.08$, and $\nu_4 = 0.09$.

In a first example, we took $N_\mathrm{f} = 2$ with $\nu_0 = 0.15$ and $\nu_1 = 0.18$. Figure 1 compares the two algorithms. For the 'cohere' function, a Hanning window of length 100 was chosen with 50% overlap, while for the proposed algorithm the parameter settings were $K = L = 100$ and the correlation matrices were estimated (with 1024 samples) by simple averaging. Clearly, the proposed algorithm is much closer to the theoretical values of the MSC function than the 'cohere' function.

In a second example, we increased the number of coherent frequencies to $N_\mathrm{f} = 5$ with $\nu_0 = 0.05$, $\nu_1 = 0.06$, $\nu_2 = 0.07$, $\nu_3 = 0.08$, and $\nu_4 = 0.09$. Figure 2 compares again the two algorithms with the same parameters of Fig. 1. The MVDR MSC is still much better. In order to increase the resolution, we augmented the window length of the 'cohere' function to 200 and took $K = 200$ for the new algorithm. Figure 3(b) (proposed algorithm) shows very clearly the 5 expected peaks while Fig. 3(a) ('cohere' function) does not have this ability.

5. CONCLUSIONS

The coherence function plays a major role in a huge number of applications. In spite of its importance, not so many algorithms exist in the literature to estimate it correctly. The most popular approach to do so is based on the Welch's method, which does not give, in our opinion, satisfactory results. Even though the MVDR principle is very popular in array processing, it was certainly underestimated for spectral estimation. In this paper, we have shown for the first time, that the MVDR concept can be easily extended for the estimation of the MSC. Many simulations show very clearly the superiority of the new algorithm over the one based on the Welch's method.

6. REFERENCES

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