# A Widely Linear Distortionless Filter for Single-Channel Noise Reduction

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***Abstract*—Traditionally in the single-channel noise-reduction problem, speech distortion is inevitable since the desired signal is also filtered while filtering the noise. In fact, the more the noise is reduced, the more the speech distortion is added into the desired signal, as proved in the literature. So, if we require no speech distortion, we either end up with no noise reduction at all or have to use multiple sensors. In this paper, we attempt to apply the widely linear (WL) estimation theory to noise reduction. Unlike the traditional approaches that only filter the short-time Fourier transform (STFT) of the noisy signal, the method developed in this paper applies the noise-reduction filter to both the STFT of the noisy signal and its conjugate. With the constraint of no speech distortion, a WL distortionless filter is derived. We show that this new optimal filter can fully take advantage of the noncircularity property of speech signals to achieve up to 3-dB signal-to-noise-ratio (SNR) improvement without introducing any speech distortion, which can only be obtained with the traditional approaches if two or more microphones are used.**

***Index Terms*—Distortionless filter, noise reduction, noncircularity, speech enhancement, widely linear filter.**

## I. Introduction

TYPICALLY, single-channel noise reduction is formulated as a digital filtering problem. In such a formulation, the core issue is to design an optimal filter that can fully exploit the speech and noise statistics to achieve maximum noise suppression without introducing perceptually noticeable speech distortion. While the optimal filters can be designed in the time domain, most widely used approaches so far work in the frequency domain. When we work in the frequency domain, we generally deal with complex random variables even though the original time-domain signals are real in the context of speech applications. For a zero-mean complex random variable (CRV), there are two basic types of second-order statistics depending on whether the random variable is circular or noncircular.

A CRV $A$ is said to be circular if its probability density function (PDF) is the same as the PDF of $Ae^{jr}$ [1]–[3], where $j$ and $r$ are the imaginary unit ($j^2 = -1$) and any real number, respectively. This is equivalent to saying that the PDF of a circular CRV (CCRV) is a function of the product $AA^*$ only [1], where $^*$ denotes complex conjugation. An important consequence of this is that the only nonnull moments and cumulants of a CCRV are the moments and cumulants constructed with the same power of $A$ and $A^*$ [1]. Now let us confine our discussion and study to the second-order issues. With the general definition of circularity, we can readily define the second-order circularity: a zero-mean complex random variable $A$ is said to be second-order circular if its pseudo-variance is equal to zero, i.e., $E\left(A^2\right) = 0$, where $E(\cdot)$ denotes mathematical expectation and $E\left(AA^*\right) = E\left(|A|^2\right) \neq 0$. This indicates that the second-order behavior of a CCRV is well described by its variance. Note that the Fourier components of stationary signals are CCRVs [4].

However, the STFT coefficients of a nonstationary signal like speech are not circular variables, as shown in [5], [6]. So a critical question one may ask is how to apply the speech noncircularity to the noise-reduction problem. In [5], [6] we developed a widely linear (WL) Wiener filter and demonstrated that the WL Wiener filter is superior to the classical Wiener filter for noise reduction. In this paper, we attempt to develop a distortionless filter. We show that this new single-channel optimal filter can achieve up to 3-dB SNR improvement without introducing any speech distortion, which can only be accomplished with the traditional techniques if two or more microphones are used.

## II. Problem Formulation

The noise-reduction problem considered in this paper is one of recovering the nonstationary desired signal (clean speech) $x(k)$, $k$ being the discrete-time index, of zero mean from the noisy observation (microphone signal):

$$y(k) = x(k) + v(k) \tag{1}$$

where $v(k)$ is the unwanted additive noise, which is assumed to be a zero-mean random process (white or colored, stationary or not) and uncorrelated with $x(k)$. In the STFT domain, (1) can be rewritten as

$$Y(n,m) = X(n,m) + V(n,m) \tag{2}$$

where $Y(n,m)$, $X(n,m)$, and $V(n,m)$ are respectively the STFTs of $y(k)$, $x(k)$, and $v(k)$, at time-frame $n$ and frequency-bin $m$ (with $m = 0, 1, \ldots, M-1$).

Using the fact that $x(k)$ and $v(k)$ are assumed to be uncorrelated, we can write the variance of the noisy spectral coefficients as

$$\phi_y(n,m) = \phi_x(n,m) + \phi_v(n,m) \tag{3}$$

where

$$\phi_a(n,m) \triangleq E\left[\left|A(n,m)\right|^2\right] \tag{4}$$

is the variance of $A(n,m)$; $A(n,m)$ is the STFT coefficients of the signal $a(k)$ at time-frame $n$ and frequency-bin $m$, and $a \in \{x, v, y\}$.

With the signal model given in (2), the noise-reduction problem becomes one of estimating $X(n,m)$ given $Y(n,m)$.

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In the traditional approaches, an estimate of $X(n,m)$ is obtained by applying a filter to $Y(n,m)$. In this paper, we consider estimating $X(n,m)$ using the WL estimation technique [7] by assuming that $Y(n,m)$ is complex[1]:

$$\begin{aligned} Z(n,m) &= H(n,m)Y(n,m) + H'(n,m)Y^*(n,m) \\ &= \mathbf{h}^H(n,m)\mathbf{y}(n,m) \end{aligned} \tag{5}$$

where $Z(n,m)$ is the STFT of the signal $z(k)$ [which is an estimate of $x(k)$], $H(n,m)$ and $H'(n,m)$ are two complex gains, superscript $^H$ denotes transpose conjugate, and

$$\mathbf{h}(n,m) \triangleq \begin{bmatrix} H^*(n,m) \\ H'^*(n,m) \end{bmatrix},$$
$$\mathbf{y}(n,m) \triangleq \begin{bmatrix} Y(n,m) \\ Y^*(n,m) \end{bmatrix}.$$

If $H'(n,m) = 0$ for any $n$ and $m$, (5) degenerates to the classical linear estimation theory . This, however, will not happen in general for noncircular CRVs.

The clean signal estimate obtained using (5) consists of contributions from both the desired signal and the unwanted noise. To clearly view this, let us decompose the signal $Z(n,m)$ into the following form:

$$Z(n,m) = X_{\mathrm{f}}(n,m) + V_{\mathrm{rn}}(n,m) \tag{6}$$

where $X_{\mathrm{f}}(n,m) \triangleq \mathbf{h}^H(n,m)\mathbf{x}(n,m)$ is the filtered version of the desired signal and its complex conjugate [where $\mathbf{x}(n,m)$ is defined in a similar way to $\mathbf{y}(n,m)$], and $V_{\mathrm{rn}}(n,m) \triangleq \mathbf{h}^H(n,m)\mathbf{v}(n,m)$ is the residual noise, which is apparently uncorrelated with $X_{\mathrm{f}}(n,m)$.

From (6), we deduce the variance of the spectral coefficients of the signal $z(k)$ at time-frame $n$ and frequency-bin $m$:

$$\phi_z(n,m) = \phi_{x_{\mathrm{f}}}(n,m) + \phi_{v_{\mathrm{rn}}}(n,m) \tag{7}$$

where

$$\begin{aligned} \phi_{x_{\mathrm{f}}}(n,m) &\triangleq E\left[|X_{\mathrm{f}}(n,m)|^2\right] \\ &= \mathbf{h}^H(n,m)\mathbf{\Phi_x}(n,m)\mathbf{h}(n,m) \end{aligned} \tag{8}$$
$$\begin{aligned} \phi_{v_{\mathrm{rn}}}(n,m) &\triangleq E\left[|V_{\mathrm{rn}}(n,m)|^2\right] \\ &= \mathbf{h}^H(n,m)\mathbf{\Phi_v}(n,m)\mathbf{h}(n,m) \end{aligned} \tag{9}$$

and

$$\begin{aligned} \mathbf{\Phi_a}(n,m) &\triangleq E\left[\mathbf{a}(n,m)\mathbf{a}^H(n,m)\right] \\ &= \phi_a(n,m)\begin{bmatrix} 1 & \gamma_a(n,m) \\ \gamma_a^*(n,m) & 1 \end{bmatrix} \\ &= \phi_a(n,m)\mathbf{\Gamma_a}(n,m) \end{aligned} \tag{10}$$

is the covariance matrix of $\mathbf{a}(n,m) = [A(n,m)\ A^*(n,m)]^T$ with

$$\gamma_a(n,m) \triangleq \frac{E\left[A^2(n,m)\right]}{E\left[|A(n,m)|^2\right]} \tag{11}$$

being the (second-order) circularity quotient [8], and $\mathbf{\Gamma_a}(n,m)$ being the circularity matrix. It can easily be shown that [8]

$$0 \le |\gamma_a(n,m)| \le 1. \tag{12}$$

The circularity coefficient $|\gamma_a(n,m)|$ conveys information about the degree of circularity of the signal $A(n,m)$. In particular, if $A(n,m)$ is a (second-order) CCRV then $\gamma_a(n,m) = 0$ and $\mathbf{\Gamma_a}(n,m) = \mathbf{I}$, where

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [\mathbf{i}_1 \quad \mathbf{i}_2] \tag{13}$$

is the $2 \times 2$ identity matrix.

The signal $X_{\mathrm{f}}(n,m)$ consists of components from both the desired signal $X(n,m)$ and its conjugate. But not all these components are what we want (this is different from the classical linear filtering problem). It is, therefore, necessary and important to distinguish between the filtered desired signal and the residual interference that both may exist in $X_{\mathrm{f}}(n,m)$ at the same time. Specifically, $H(n,m)X(n,m)$ is part of the overall filtered desired signal, but $H'(n,m)X^*(n,m)$ is not. If $\gamma_x(n,m) = 0$ for any $n$ and $m$, $X(n,m)$ and $X^*(n,m)$ are uncorrelated and the overall filtered desired signal is indeed $H(n,m)X(n,m)$. But for $\gamma_x(n,m) \ne 0$, $X^*(n,m)$ is correlated with $X(n,m)$ and contains both the desired signal and an interference component. Following the idea developed in [5], [9], we can decompose $X^*(n,m)$ into two orthogonal components:

$$X^*(n,m) = \gamma_x^*(n,m)X(n,m) + X'(n,m) \tag{14}$$

where

$$X'(n,m) = X^*(n,m) - \gamma_x^*(n,m)X(n,m), \tag{15}$$
$$E\left[X(n,m)X'^*(n,m)\right] = 0 \tag{16}$$

and

$$E\left[|X'(n,m)|^2\right] = \phi_x(n,m)\left[1 - |\gamma_x(n,m)|^2\right]. \tag{17}$$

We can then rewrite (6) as

$$Z(n,m) = X_{\mathrm{fd}}(n,m) + X'_{\mathrm{ri}}(n,m) + V_{\mathrm{rn}}(n,m) \tag{18}$$

where

$$\begin{aligned} X_{\mathrm{fd}}(n,m) &\triangleq \mathbf{h}^H(n,m)\mathbf{\Gamma_x}(n,m)\mathbf{i}_1 X(n,m) \\ &= H(n,m)X(n,m) \\ &\quad + \gamma_x^*(n,m)H'(n,m)X(n,m), \end{aligned} \tag{19a}$$
$$X'_{\mathrm{ri}}(n,m) \triangleq \mathbf{h}^H(n,m)\mathbf{i}_2 X'(n,m), \tag{19b}$$
$$V_{\mathrm{rn}}(n,m) \triangleq \mathbf{h}^H(n,m)\mathbf{v}(n,m) \tag{19c}$$

are the overall filtered desired signal, the residual interference, and the residual additive noise, respectively. Note that the above decomposition of the signal $X^*(n,m)$ is critical in order to properly design the optimal WL noise-reduction filters.

The three terms on the right-hand side of (18) are mutually uncorrelated. Therefore, we have

$$\phi_z(n,m) = \phi_{x_{\mathrm{fd}}}(n,m) + \phi_{x'_{\mathrm{ri}}}(n,m) + \phi_{v_{\mathrm{rn}}}(n,m) \tag{20}$$

[1]If $Y(n,m)$ is real, the WL estimator is the same as that using the classical techniques, which has already been covered in the rich literature.

where

$$\begin{aligned}\phi_{x_{\mathrm{fd}}}(n,m) &\triangleq E\left[|X_{\mathrm{fd}}(n,m)|^2\right] \\ &= \phi_x(n,m)\mathbf{h}^H(n,m)\mathbf{\Gamma_x}(n,m) \\ &\quad \times \mathbf{i}_1\mathbf{i}_1^H\mathbf{\Gamma_x}(n,m)\mathbf{h}(n,m), \end{aligned} \tag{21}$$

$$\begin{aligned}\phi_{x'_{\mathrm{ri}}}(n,m) &\triangleq E\left[|X'_{\mathrm{ri}}(n,m)|^2\right] \\ &= \phi_x(n,m)\left[1-|\gamma_x(n,m)|^2\right] \\ &\quad \times \mathbf{h}^H(n,m)\mathbf{i}_2\mathbf{i}_2^H\mathbf{h}(n,m)\end{aligned} \tag{22}$$

and $\phi_{v_{\mathrm{rn}}}(n,m)$ is defined in (9).

Now, we define the subband error signal between the estimated and desired signals as

$$\begin{aligned}\mathcal{E}(n,m) &\triangleq Z(n,m) - X(n,m) \\ &= \mathbf{h}^H(n,m)\mathbf{y}(n,m) - X(n,m)\end{aligned} \tag{23}$$

which can be written as the sum of two error signals:

$$\mathcal{E}(n,m) = \mathcal{E}_{\mathrm{d}}(n,m) + \mathcal{E}_{\mathrm{r}}(n,m) \tag{24}$$

where

$$\begin{aligned}\mathcal{E}_{\mathrm{d}}(n,m) &\triangleq X_{\mathrm{fd}}(n,m) - X(n,m) \\ &= \left[\mathbf{h}^H(n,m)\mathbf{\Gamma_x}(n,m)\mathbf{i}_1 - 1\right]X(n,m)\end{aligned} \tag{25}$$

is the signal distortion due to the complex filter and

$$\begin{aligned}\mathcal{E}_{\mathrm{r}}(n,m) &\triangleq X'_{\mathrm{ri}}(n,m) + V_{\mathrm{rn}}(n,m) \\ &= \mathbf{h}^H(n,m)\mathbf{i}_2X'(n,m) \\ &\quad + \mathbf{h}^H(n,m)\mathbf{v}(n,m)\end{aligned} \tag{26}$$

represents the residual interference and noise.

The subband mean-squared error (MSE) is then

$$J\left[\mathbf{h}(n,m)\right] = J_{\mathrm{d}}\left[\mathbf{h}(n,m)\right] + J_{\mathrm{r}}\left[\mathbf{h}(n,m)\right] \tag{27}$$

where

$$\begin{aligned}J_{\mathrm{d}}\left[\mathbf{h}(n,m)\right] &= E\left[|X_{\mathrm{fd}}(n,m) - X(n,m)|^2\right] \\ &= \phi_x(n,m)\left|\mathbf{h}^H(n,m)\mathbf{\Gamma_x}(n,m)\mathbf{i}_1 - 1\right|^2\end{aligned} \tag{28}$$

and

$$\begin{aligned}J_{\mathrm{r}}\left[\mathbf{h}(n,m)\right] &= E\left[|X'_{\mathrm{ri}}(n,m)|^2\right] + E\left[|V_{\mathrm{rn}}(n,m)|^2\right] \\ &= \phi_{x'_{\mathrm{ri}}}(n,m) + \phi_{v_{\mathrm{rn}}}(n,m).\end{aligned} \tag{29}$$

It is clear that the objective of noise reduction in the frequency domain is to find optimal gains $H(n,m)$ and $H'(n,m)$ at each time-frame $n$ and frequency-bin $m$ that would either directly minimize $J\left[\mathbf{h}(n,m)\right]$ or minimize $J_{\mathrm{d}}\left[\mathbf{h}(n,m)\right]$ or $J_{\mathrm{r}}\left[\mathbf{h}(n,m)\right]$ subject to some constraint.

## III. Widely Linear Distortionless Filter

Having defined the subband MSE, we are now ready to derive noise-reduction filters. As a matter of fact, minimizing $J\left[\mathbf{h}(n,m)\right]$ with respect to $\mathbf{h}(n,m)$ leads to the WL Wiener filter, which was shown to outperform the classical Wiener filter for noise reduction [5]. However, like the classical approaches where noise reduction is achieved always by adding distortion to the desired signal, the WL Wiener filter also introduces some speech distortion. In this section, we show that it is possible to derive a WL distortionless filter whose noise-reduction performance depends exclusively on the noncircularities of the desired and noise signals.

From (25), we see that the constraint to avoid any distortion on the desired signal is

$$\mathbf{h}^H(n,m)\mathbf{\Gamma_x}(n,m)\mathbf{i}_1 = 1. \tag{30}$$

Therefore, minimizing the MSE, $J\left[\mathbf{h}(n,m)\right]$, subject to the constraint (30) would lead to a WL distortionless noise-reduction filter. This is also equivalent to finding a filter that minimizes $E\left[|Z(n,m)|^2\right]$ subject to the constraint (30), i.e., see (31) at the bottom of the page. If we use a Lagrange multiplier to adjoin the constraint to the cost function and then equating the derivative of the cost function with respect to $\mathbf{h}(n,m)$ to zero, we readily derive the WL distortionless filter:

$$\begin{aligned}\mathbf{h}_{\mathrm{WLD}}(n,m) &= \left[\mathbf{i}_1^H\mathbf{\Gamma_x}(n,m)\mathbf{\Phi_y}^{-1}(n,m)\mathbf{\Gamma_x}(n,m)\mathbf{i}_1\right]^{-1} \\ &\quad \times \mathbf{\Phi_y}^{-1}(n,m)\mathbf{\Gamma_x}(n,m)\mathbf{i}_1, \end{aligned} \tag{32a}$$

$$\begin{aligned}&= \left[\mathbf{i}_1^H\mathbf{\Gamma_x}(n,m)\mathbf{\Gamma_y}^{-1}(n,m)\mathbf{\Gamma_x}(n,m)\mathbf{i}_1\right]^{-1} \\ &\quad \times \mathbf{\Gamma_y}^{-1}(n,m)\mathbf{\Gamma_x}(n,m)\mathbf{i}_1, \end{aligned} \tag{32b}$$

$$\begin{aligned}&= \left[\mathbf{i}_1^H\mathbf{\Gamma_x}(n,m)\mathbf{\Phi}_{\mathrm{in}}^{-1}(n,m)\mathbf{\Gamma_x}(n,m)\mathbf{i}_1\right]^{-1} \\ &\quad \times \mathbf{\Phi}_{\mathrm{in}}^{-1}(n,m)\mathbf{\Gamma_x}(n,m)\mathbf{i}_1\end{aligned} \tag{32c}$$

where

$$\mathbf{\Phi}_{\mathrm{in}}(n,m) = \phi_x(n,m)\left[1-|\gamma_x(n,m)|^2\right]\mathbf{i}_2\mathbf{i}_2^H + \mathbf{\Phi_v}(n,m) \tag{33}$$

is the covariance matrix of the interference plus noise. If the desired and noise signals are circular, i.e., $\mathbf{\Gamma_x}(n,m) = \mathbf{\Gamma_v}(n,m) = \mathbf{I}$, we get $\mathbf{h}_{\mathrm{WLD}}(n,m) = \mathbf{i}_1$, which is the classical distortionless filter; this particular filter does not, of course, distort the desired signal but it does not reduce the noise either. However, when the desired and noise signals are not circular, their noncircularity can help achieve noise reduction without adding speech distortion. To verify this, we give the following property.

$$\min_{\mathbf{h}(n,m)} \; E\left[|Z(n,m)|^2\right] = \min_{\mathbf{h}(n,m)} \mathbf{h}^H(n,m)\mathbf{\Phi_y}(n,m)\mathbf{h}(n,m) \text{ subject to } \mathbf{h}^H(n,m)\mathbf{\Gamma_x}(n,m)\mathbf{i}_1 = 1. \tag{31}$$

*Property:* With the WL distortionless filter given in (32), the subband output SNR is always greater than or equal to the subband input SNR, i.e., $\mathrm{oSNR}\left[\mathbf{h}_{\mathrm{WLD}}(n,m)\right] \geq \mathrm{iSNR}(n,m) \triangleq \phi_x(n,m)/\phi_v(n,m),\ \forall n,m$.

*Proof:* Indeed, the subband output SNR is shown in (34) at the bottom of the page. With some simple mathematical manipulation on the inverse of the $2 \times 2$ matrix $\mathbf{\Phi}_{\mathrm{in}}(n,m)$, we easily find the SNR gain, which is shown in (35) at the bottom of the page, with equality if and only if $\gamma_x(n,m) = \gamma_v(n,m) = 0$ (i.e., circular signals). It follows immediately that $\mathrm{oSNR}\left[\mathbf{h}_{\mathrm{WLD}}(n,m)\right] \geq \mathrm{iSNR}(n,m),\ \forall n,m$.

If noise is stationary, which is often assumed to be the case in practical situations, one can see that the SNR gain $\eta(n,m)$ would only depend on the input SNR, i.e., $\mathrm{iSNR}(n,m)$, and the speech noncircularity parameter $|\gamma_x(n,m)|^2$. Fig. 1 plots the SNR gain as a function of $\mathrm{iSNR}(n,m)$ and $|\gamma_x(n,m)|^2$. One can see that up to 3-dB SNR improvement can be achieved if the desired signal is noncircular and the input SNR is relatively low. It is quite remarkable that it is possible to design a distortionless filter that fully exploits the noncircularities of the signals to improve the subband SNR (even though the amount of the SNR improvement may not be significant), while the traditional techniques have no effect on the subband SNR, i.e., $\mathrm{oSNR}\left[\mathbf{h}(n,m)\right]$ is always equal to $\mathrm{iSNR}(n,m)$ . With the new WL distortionless filter, if we want to achieve more than 3-dB SNR gain, this can be obtained by filtering cross-frame spectra, which is currently under investigation.

Many simulations with speech signals have shown, at this point of time, that we can get up to 1-dB improvement in SNR without adding distortion to the desired signal. However, because of space limitation the results are not presented here.

## IV. CONCLUSIONS

The traditional single-channel noise-reduction techniques achieve noise reduction by paying a price of speech distortion; and the more the noise is reduced, the more the speech distortion is added into the desired signal. So, with the traditional techniques, if we require no speech distortion, we would end up with no noise reduction. In this paper, we developed a WL distortionless filter for single-channel noise reduction based on the WL estimation theory. We showed that this new optimal filter can fully take advantage of the noncircularity property of the speech and noise signals to achieve noise reduction without introducing any speech distortion. If noise is stationary, the WL distortionless filter can achieve up to 3-dB SNR improvement, which would need two or more microphones to obtain with the classical linear estimation theory.

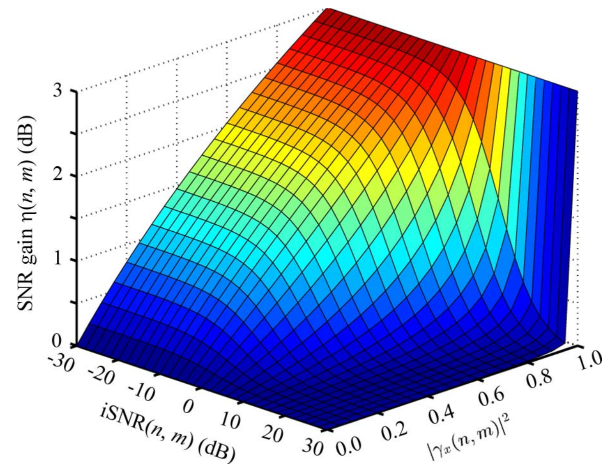


Fig. 1. Theoretical SNR gain of the WL distortionless filter as a function of $|\gamma_x(n,m)|^2$ and $\mathrm{iSNR}(n,m)$.

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$$\mathrm{oSNR}\left[\mathbf{h}_{\mathrm{WLD}}(n,m)\right] = \frac{\phi_x(n,m)}{\mathbf{h}_{\mathrm{WLD}}^H(n,m)\mathbf{\Phi}_{\mathrm{in}}(n,m)\mathbf{h}_{\mathrm{WLD}}(n,m)} = \phi_x(n,m) \cdot \mathbf{i}_1^H \mathbf{\Gamma}_{\mathbf{x}}(n,m)\mathbf{\Phi}_{\mathrm{in}}^{-1}(n,m)\mathbf{\Gamma}_{\mathbf{x}}(n,m)\mathbf{i}_1. \tag{34}$$

$$\eta(n,m) = \frac{\mathrm{oSNR}\left[\mathbf{h}_{\mathrm{WLD}}(n,m)\right]}{\mathrm{iSNR}(n,m)} = 1 + \frac{|\gamma_x(n,m) - \gamma_v(n,m)|^2}{1 - |\gamma_v(n,m)|^2 + \mathrm{iSNR}(n,m)\left[1 - |\gamma_x(n,m)|^2\right]} \geq 1 \tag{35}$$